On two-dimensional directed percolation

1988 J. Phys. A: Math. Gen. 21 3815

(http://iopscience.iop.org/0305-4470/21/19/018)

J. Phys. A: Math. Gen. 21 (1988) 3815-3832. Printed in the UK

On two-dimensional directed percolation

J W Essam†, A J Guttmann‡ and K De'Bell§

† Department of Mathematics, Royal Holloway and Bedford New College, University of London, Egham Hill, Egham, Surrey TW20 0EX, UK
‡ Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia
§ Department of Physics, Dalhousie University, Halifax, Nova Scotia B3H 3J5, Canada

Received 18 March 1988

Abstract. Extended series expansions for the mean size and the first and second moments of the pair connectedness for both bond and site percolation on the directed square and triangular lattices have been obtained. Analysis based on differential approximants allows the critical percolation probabilities and exponents to be estimated, and as a result the critical exponents are conjectured to be $\gamma = 41/18$, $\nu_\perp = 79/72$ and $\nu_\| = 26/15$. Scaling then gives $\beta = 199/720$, $\alpha = -299/360$ and $\delta = 1839/199$.

1. Introduction

In an earlier paper (Essam *et al* 1986, hereafter referred to as I), we reported on an analysis of the first 35 terms of the moments of the pair connectedness $C_i(p)$ for bond percolation on the directed square lattice. The moments studied are defined as follows:

$$S = \mu_0 = \sum C_i(p) \qquad \mu_{2x} = \langle x^2 \rangle = \sum x_i^2 C_i(p) \qquad \mu_{2t} = \langle t^2 \rangle = \sum t_i^2 C_i(p) \tag{1.1}$$

where x_i and t_i are the coordinates of the ith lattice site perpendicular and parallel to the preferred (1, 1) direction. The zeroth moment is the mean-size series, and its critical exponent is usually denoted γ. The two second-moment series have exponents $\gamma + 2\nu_\perp$ and $\gamma + 2\nu_\|$ respectively.

These series were generated and extended by supplementing the transfer-matrix method of Blease (1977) with a weak subgraph expansion (De'Bell and Essam 1983) and extended using a Dyson equation. Further details are provided in § 2.

In I, the series for the square lattice bond percolation problem were investigated using standard Padé methods and the method of Adler *et al* (1981) which is designed to reveal and identify confluent exponents. While these methods, combined with the longer series, gave considerably improved exponent and percolation probability estimates, the claimed errors were perhaps optimistic, as the methods of analysis could not cope with certain functional features believed to be present in the moment series, such as an additive analytic background term. To illustrate this point, if one takes a series, and performs a Dlog Padé analysis on the series, certain exponent and critical point estimates will emerge. If one then changes the first term in the series by one, say, repetition of the analysis gives rise to a slightly different set of estimates. Such a change simulates a background term. The method of differential approximants (Guttmann and Joyce 1972, Joyce and Guttmann 1973, Rehr *et al* 1980) can accommodate such changes, as well as confluent singularities, logarithmic divergences and most of the functional features expected in non-pathological models of statistical mechanical

0305-4470/88/193815+18$02.50

systems. In § 3, we analyse the new series, which have been substantially extended. We also analyse the new first-moment series, $\mu_{1t} = \langle t \rangle = \Sigma\, t_i C_i(p)$.

Recent understanding of two-dimensional lattice models has led to the belief that critical exponents for such systems should be simple rational fractions. Such a conclusion follows from conformal invariance theory (Cardy 1987) in which case the various operators are quantised, giving rise to rational exponents.

Conformal invariance theory requires that the correlation functions be invariant under translation. In the problem of directed percolation all correlation functions are defined relative to a particular source and (once the source has been chosen) translational invariance is completely destroyed. However, there is still rotational invariance about the axis through the source parallel to the special direction.

Nevertheless, we examine the possibility that rational exponents will be found even in this case. Indeed, if we look at the exponent estimates in I, which were $\gamma = 2.277\,21 \pm 0.0003$, $\nu_\perp = 1.0972 \pm 0.0006$ and $\nu_\| = 1.7334 \pm 0.001$ (with additional uncertainties proportional to the error in p_c) and seek the most 'obvious' exact fractions, then $\gamma = 41/18 = 2.277\,777\ldots$, $\nu_\perp = 79/72 = 1.097\,2222\ldots$ and $\nu_\| = 26/15 = 1.733\,333$ suggest themselves. The value for γ is particularly appealing as the value of the corresponding exponent for ordinary percolation is $\gamma = 43/18 = 2.388\,88\ldots$, while the correlation function exponent $\nu_\perp$ is less convincing, with its large denominator. It must be remembered that it is the directed nature of this problem that gives rise to two distinct correlation exponents, and so any non-simple aspect of the problem might be expected to manifest itself in the value of the correlation function exponents. The scaling law $\beta = (\nu_\| + \nu_\perp - \gamma)/2$ then gives $\beta = 199/720$, while the scaling law $\alpha + 2\beta + \gamma = 2$ gives $\alpha = -299/360$. Both these values have unusually large denominators, while increasing the numerators just by 1 yields the far simpler and certainly more appealing fractions $\beta = 5/18$ and $\alpha = -5/6$. These would imply that $\nu_\| + \nu_\perp = 17/6$. In § 3 we argue that the numerical evidence favours the former exponent set. In § 4 we return to the question of conformal invariance and point out that our conjectured exponent values do not correspond to any single family of exponents, as characterised by a particular central change.

2. Derivation of series expansions

Low-density series expansions for the mean cluster size and spatial moments of directed lattice percolation models have previously been obtained using a transfer-matrix method for the pair connectedness (Blease 1977, De'Bell and Essam 1983). Here we show that the same transfer-matrix method used in conjunction with non-nodal graph expansions allows the length of the series obtained by the basic transfer-matrix method to be doubled.

2.1. Non-nodal graph expansions

Let $S(t)$ be the expected number of sites which are connected to the origin and whose distance from the origin measured parallel to the preferred direction is t. In terms of $S(t)$ the mean cluster size and first two parallel moments of the cluster mass distribution are given, respectively, by

$$S = \sum_{t=0}^{\infty} S(t) \qquad \mu_{1t} = \sum_{t=1}^{\infty} tS(t) \qquad \mu_{2t} = \sum_{t=1}^{\infty} t^2 S(t) \tag{2.1}$$

where the dependence on p has been suppressed and $S(0)=1$. The function $S(t)$ is related to the pair connectedness $C(x, t)$ by

$$S(t)=\sum_{x} C(x, t) \tag{2.2}$$

where the sum is over all lattice sites whose parallel distance from the origin is t and the vector x is the component of the position vector of a given such site perpendicular to the preferred direction. The function

$$X(t)=\sum_{x} x^2 C(x, t) \tag{2.3}$$

will also be considered and serves to determine the second perpendicular moment of the cluster mass distribution

$$\mu_{2x}=\sum_{t=0}^{\infty} X(t). \tag{2.4}$$

The pair connectedness $C(x, t)$ may be expressed (Essam 1972) as a sum over all subgraphs of the lattice graph which may be formed by taking unions of possible directed paths connecting the origin to the site (x, t):

$$C(x, t)=\sum_{g} d(g)p^e \tag{2.5}$$

where e is the number of random elements (sites or bonds) in g and in the case of site percolation the site at the origin, which is the source, is not counted as a random element. A graph g is nodal if there is an intermediate vertex through which all the above-mentioned paths must pass. This vertex is called a nodal point. The non-nodal contribution $S^{\mathrm{N}}(t)$ to $S(t)$ is defined by the above sum over graphs (2.5) restricted to non-nodal graphs. By convention $S^{\mathrm{N}}(0)=0$. If g is the series combination of graphs g_1 and g_2, so that their common vertex is a nodal point, then the d weight $d(g)$ factorises as the product of the d weights for the two separate graphs. This was used by Bhatti and Essam (1984) to show that S satisfies a 'Dyson equation':

$$S=1+S^{\mathrm{N}}S \tag{2.6}$$

where, here and below, the superscript N denotes that $S(t)$ has been replaced by $S^{\mathrm{N}}(t)$ in this case in the definition (2.1) of S, and following the derivation of Bhatti and Essam we obtain, for $t \geqslant 1$:

$$S(t)=\sum_{t'=1}^{t} S^{\mathrm{N}}(t')S(t-t') \tag{2.7}$$

from which (2.6) follows by summation over t.

Using the definition of μ_{1t} (2.1) together with (2.7)

$$\begin{aligned}\mu_{1t}&=\sum_{t=1}^{\infty} t \sum_{t'=1}^{t} S^{\mathrm{N}}(t')S(t-t')\\ &=\sum_{t'=1}^{\infty}\sum_{t''=0}^{\infty}(t'+t'')S^{\mathrm{N}}(t')S(t'') \qquad (2.8)\\ &=\mu_{1t}^{\mathrm{N}}S+S^{\mathrm{N}}\mu_{1t}. \qquad (2.9)\end{aligned}$$

Combining (2.9) and (2.6)

$$\mu_{1t}=\mu_{1t}^{\mathrm{N}}S^2. \tag{2.10}$$

Similarly, replacing $(t'+t'')$ by $(t'+t'')^2$ in (2.8)

$$\mu_{2t} = \mu_{2t}^{\mathrm{N}} S + 2\mu_{1t}^{\mathrm{N}}\mu_{1t} + S^{\mathrm{N}}\mu_{2t} = [\mu_{2t}^{\mathrm{N}} + 2(\mu_{1t}^{\mathrm{N}})^2 S]S^2. \tag{2.11}$$

The corresponding relation for μ_{2x} may be obtained by substituting (2.5) into (2.3) and then following Bhatti and Essam's derivation of (2.6) with the result, for $t \geq 1$:

$$X(t) = \sum_{t'=1}^{t} [S^{\mathrm{N}}(t')X(t-t') + X^{\mathrm{N}}(t')S(t-t')] \tag{2.12}$$

where we have assumed that the symmetry of the lattice is such that the first perpendicular moment of the cluster mass distribution, restricted to atoms with coordinate t, is zero. Notice that $X(0) = 0$ and by convention $X^{\mathrm{N}}(0) = 0$. Summing over t and using (2.4) and (2.6), we obtain

$$\mu_{2x} = \mu_{2x}^{\mathrm{N}} S^2. \tag{2.13}$$

2.2. *Series expansion algorithm*

In a previous paper it was shown how $S(t)$ and $X(t)$ could be obtained by t iterations of a transfer matrix. These functions are polynomials in p and from (2.5) it follows that the powers of p less than $m(t)$ are zero, where $m(t)$ is the length of the shortest walk required to reach a site whose parallel distance from the origin is t. For the square lattice $m(t) = t$ but for the triangular lattice $m(t) = [(t+1)/2]$, where [ ] denotes integer below. Therefore if $S(t')$ and $X(t')$ are determined for $t \leq t$ to order $m(t+1)-1$ then the mean size and moments will be determined to order $m(t+1)-1$. For $t \geq 2$ the functions $S^{\mathrm{N}}(t)$ and $X^{\mathrm{N}}(t)$ are polynomials, the leading power of p of which is determined by the smallest number of random elements $n(t)$ which are needed to provide two parallel paths, the intermediate vertices of which are disjoint. For bond percolation on the square lattice $n(t) = 2t$ and for the triangular lattice $n(t) = t+1$. In the case of site percolation $n(t)$ is one less than for bond percolation since both paths have the same terminal vertex (the initial vertex is considered to be non-random). In any case $n(t)$ is approximately $2m(t)$ which is the key to the following improved algorithm. The steps are as follows.

(i) Use the transfer-matrix method to obtain the polynomials $S(t')$ and $X(t')$ for $t' \leq t$ to order $n(t+1)-1$ (rather than $m(t+1)-1$ as in the standard method).

(ii) Set $S^{\mathrm{N}}(1) = S(1)$ and $X^{\mathrm{N}}(1) = X(1)$.

(iii) For $2 \leq t' \leq t$ use the recurrence formulae

$$S^{\mathrm{N}}(t') = S(t') - \sum_{t''=1}^{t'-1} S^{\mathrm{N}}(t'')S(t'-t'') \tag{2.14}$$

and

$$X^{\mathrm{N}}(t') = X(t') - \sum_{t''=1}^{t'-1} [S^{\mathrm{N}}(t'')X(t'-t'') + X^{\mathrm{N}}(t'')S(t'-t'')] \tag{2.15}$$

to determine $S^{\mathrm{N}}(t')$ and $X^{\mathrm{N}}(t')$ correct to order $n(t+1)-1$. These formulae follow by rearrangement of equations (2.7) and (2.12).

(iv) Form the sums (2.1) and (2.4) as far as t, with S and X replaced by S^{N} and X^{N}, using the truncated polynomials $S^{\mathrm{N}}(t')$ and $X^{\mathrm{N}}(t')$ of (iii) to obtain S^{N}, μ_{1t}^{N}, μ_{2t}^{N} and μ_{2x}^{N} correct to order $p^{n(t+1)-1}$ (notice that the corresponding sums using $S(t)$ and $X(t)$ would only be correct to $p^{m(t+1)-1)}$).

(v) Use formulae (2.6), (2.10), (2.11) and (2.13) to obtain S, μ_{1t}, μ_{2t} and μ_{2x} correct to order $p^{n(t+1)-1}$.

We illustrate the algorithm by the following example and our results obtained by programming the algorithm are listed in table 1.

For the square lattice bond problem with $t=3$, $n(t+1)-1=7$, the transfer-matrix method could be used to obtain the following S and X polynomials:

$$S(1)=2p \qquad S(2)=4p^2-p^4 \qquad S(3)=8p^3-4p^5-2p^6+2p^7$$

$$X(1)=2p \qquad X(2)=8p^2 \qquad X(3)=24p^3-4p^5-2p^6+2p^7$$

from which we deduce

$$S^{\mathrm{N}}(1)=2p \qquad S^{\mathrm{N}}(2)=S(2)-S^{\mathrm{N}}(1)S(1)=4p^2-p^4-(2p)^2=-p^4$$

$$S^{\mathrm{N}}(3)=S(3)-S^{\mathrm{N}}(1)S(2)-S^{\mathrm{N}}(2)S(1)=-2p^6+2p^7$$

$$X^{\mathrm{N}}(1)=2p \qquad X^{\mathrm{N}}(2)=X(2)-S^{\mathrm{N}}(1)X(1)-X^{\mathrm{N}}(1)S(1)=0$$

$$X^{\mathrm{N}}(3)=X(3)-S^{\mathrm{N}}(1)X(2)-S^{\mathrm{N}}(2)X(1)-X^{\mathrm{N}}(1)S(2)-X^{\mathrm{N}}(2)S(1)=-2p^6+2p^7.$$

Notice the cancellation of the lower-order terms on conversion to non-nodal form. Now

$$S^{\mathrm{N}}=S^{\mathrm{N}}(1)+S^{\mathrm{N}}(2)+S^{\mathrm{N}}(3)+\mathrm{O}(p^8)=2p-p^4-2p^6+2p^7+\mathrm{O}(p^8)$$

$$\mu_{1t}^{\mathrm{N}}=S^{\mathrm{N}}(1)+2S^{\mathrm{N}}(2)+3S^{\mathrm{N}}(3)+\mathrm{O}(p^8)=2p-2p^4-6p^6+6p^7+\mathrm{O}(p^8)$$

$$\mu_{2t}^{\mathrm{N}}=S^{\mathrm{N}}(1)+4S^{\mathrm{N}}(2)+9S^{\mathrm{N}}(3)+\mathrm{O}(p^8)=2p-4p^4-18p^6+18p^7+\mathrm{O}(p^8)$$

$$\mu_{2x}^{\mathrm{N}}=X^{\mathrm{N}}(1)+X^{\mathrm{N}}(2)+X^{\mathrm{N}}(3)+\mathrm{O}(p^8)=2p-2p^6+2p^7+\mathrm{O}(p^8)$$

and from (2.6)

$$(1-S^{\mathrm{N}})S=(1-2p+p^4+2p^6-2p^7+\mathrm{O}(p^8))S=1$$

and hence

$$S=1+2p+4p^2+8p^3+15p^4+28p^5+50p^6+90p^7+\mathrm{O}(p^8)$$

and

$$S^2=1+4p+12p^2+32p^3+78p^4+180p^5+396p^6+\mathrm{O}(p^7).$$

Substituting the above results in (2.10), (2.11) and (2.13) gives

$$\mu_{1t}=2p+8p^2+24p^3+62p^4+148p^5+330p^6+710p^7+\mathrm{O}(p^8)$$

$$\mu_{2t}=2p+16p^2+72p^3+252p^4+764p^5+2094p^6+5362p^7+\mathrm{O}(p^8)$$

$$\mu_{2x}=2p+8p^2+24p^3+64p^4+156p^5+358p^6+786p^7+\mathrm{O}(p^8).$$

Table 1. The series expansions for the square lattice bond and site problem and for the triangular lattice bond and site problems.

n	$S(p)$	$\mu_{1t}(p)$	$\mu_{2t}(p)$	$\mu_{2x}(p)$
Square bond percolation				
0	1	0	0	0
1	2	2	2	2
2	4	8	16	8
3	8	24	72	24
4	15	62	252	64
5	28	148	764	156
6	50	330	2 094	358
7	90	710	5 362	786
8	156	1 464	12 968	1 664
9	274	2 962	30 138	3 434
10	466	5 814	67 446	6 902
11	804	11 288	147 048	13 656
12	1 348	21 406	311 940	26 464
13	2 300	40 364	649 860	50 772
14	3 804	74 570	1 325 234	95 754
15	6 450	137 602	2 668 130	179 442
16	10 547	249 088	5 278 066	331 294
17	17 784	451 868	10 346 200	609 496
18	28 826	804 766	19 977 010	1 106 106
19	48 464	1 440 580	38 329 556	2 004 852
20	77 689	2 529 686	72 546 986	3 586 874
21	130 868	4 482 584	136 785 444	6 423 028
22	207 308	7 775 166	254 596 418	11 351 274
23	350 014	13 664 146	473 093 498	20 126 538
24	548 271	23 446 020	868 060 738	35 191 190
25	931 584	40 953 840	1 593 517 724	61 883 196
26	1 433 966	69 518 842	2 887 257 826	107 179 834
27	2 469 368	120 978 656	5 246 647 808	187 216 848
28	3 725 257	203 223 692	9 400 175 212	321 395 596
29	6 510 384	352 808 860	16 935 336 776	558 468 104
30	9 590 838	586 473 542	30 035 008 322	950 702 594
31	17 192 714	1 018 405 966	53 731 142 846	1 645 491 278
32	24 357 702	1 671 890 010	94 373 684 636	2 778 049 248
33	45 428 434	2 913 173 846	167 898 005 054	4 796 424 622
34	61 388 268	4 717 224 772	292 175 943 812	8 028 750 772
35	119 938 514	8 265 261 498	517 568 220 986	13 848 760 938
36	152 169 019	13 170 191 912	892 446 666 230	22 970 545 738
37	320 596 894	23 329 646 078	1 576 771 977 102	39 658 497 294
38	366 032 458	36 355 510 686	2 692 167 518 718	65 097 995 126
39	863 591 282	65 539 706 454	4 753 002 697 538	112 763 087 618
40	863 729 021	99 432 015 478	8 030 862 823 894	182 857 632 886
41	2 341 276 788	183 391 807 808	14 191 946 028 360	318 657 133 880
42	1 916 799 026	268 568 296 956	23 698 437 327 532	509 161 094 708
43	6 556 348 906	513 870 876 498	42 048 096 233 634	896 268 945 170
44	3 755 360 368	714 234 719 598	69 196 800 976 500	1 404 966 444 256
45	18 610 776 960	1 440 359 201 368	123 705 722 616 080	2 511 592 640 496
46	6 082 131 438	1 874 047 502 574	200 105 287 694 726	3 842 293 796 974
47	53 874 179 752	4 048 390 833 688	361 799 444 980 384	7 020 605 858 496
48	1 495 903 344	4 791 576 314 698	572 522 672 837 924	10 398 622 970 264
49	164 440 159 702	11 521 319 804 730	1054 505 095 310 298	19 624 561 178 026

Table 1. (continued)

n	$S(p)$	$\mu_{1t}(p)$	$\mu_{2t}(p)$	$\mu_{2x}(p)$
Square site percolation				
0	1	0	0	0
1	2	2	2	2
2	4	8	16	8
3	7	22	68	24
4	12	52	220	60
5	20	112	608	136
6	33	228	1 520	288
7	53	442	3 526	582
8	85	832	7 756	1 132
9	133	1 516	16 302	2 138
10	210	2 720	33 172	3 940
11	322	4 754	65 378	7 114
12	505	8 264	126 224	12 632
13	759	14 000	237 600	22 080
14	1 192	23 824	441 776	38 160
15	1 748	39 318	802 820	65 056
16	2 782	66 052	1 451 932	110 172
17	3 931	106 282	2 563 356	184 032
18	6 476	177 884	4 544 304	306 968
19	8 579	277 936	7 818 078	503 650
20	15 216	469 384	13 684 784	831 408
21	17 847	703 924	22 938 278	1 340 338
22	36 761	1 225 052	39 986 208	2 201 840
23	33 612	1 718 226	64 996 080	3 479 116
24	93 961	3 203 156	114 280 984	5 733 312
25	47 282	3 974 696	177 912 196	8 814 468
26	262 987	8 551 248	322 438 072	14 772 040
27	−16 105	8 307 370	467 942 962	21 734 370
28	827 382	23 950 704	909 533 348	37 997 724
29	−571 524	13 195 606	1 162 410 740	51 650 456
30	2 936 705	72 779 892	2 614 286 452	98 952 836
31	−3 661 626	−1 798 186	2 595 422 914	115 227 474
32	11 507 775	246 605 280	7 869 393 556	266 750 996
33	−18 880 652	−165 440 790	4 348 425 126	223 323 542
34	48 169 220	935 635 244	25 625 330 524	768 153 044
35	−90 436 605	−1 166 043 794	−1 054 012 626	261 658 998
36	209 765 885	3 896 720 688	92 617 456 680	2 443 777 216
37	−421 114 926	−6 470 965 954	−64 842 762 130	−717 803 658
38	934 999 403	17 297 466 660	373 042 426 296	8 754 481 712
39	−1 940 096 836	−33 101 156 302	−478 809 964 204	−8 229 926 352
40	4 221 969 137	79 718 300 900	1 641 532 494 032	35 018 197 920
41	−8 903 758 084	−163 586 078 926	−2 793 424 377 040	−51 106 610 852
42	19 208 110 665	374 927 721 428	7 665 060 608 076	151 983 829 124
43	−40 856 793 461	−796 243 269 742	−15 002 173 968 860	−273 752 308 264
44	87 866 047 787	1 782 844 089 528	37 057 168 356 652	694 038 101 604
45	−187 795 694 858	−3 850 361 954 756	−77 725 687 530 014	−1 385 891 817 602
46	403 517 351 347	8 526 692 750 236	182 456 293 328 988	3 260 155 117 268
47	−864 759 759 311	−18 563 737 025 990	−395 779 410 517 728	−6 844 942 177 300
48	1858 291 322 498	40 898 675 755 280	906 220 153 528 224	15 539 271 241 976
Triangular bond percolation				
0	1	0	0	0
1	3	4	6	2
2	9	24	68	12

Table 1. (continued)

n	$S(p)$	$\mu_{1t}(p)$	$\mu_{2t}(p)$	$\mu_{2x}(p)$
Triangular bond percolation				
3	25	104	442	54
4	66	384	2 218	206
5	168	1 284	9 528	712
6	417	4 012	36 834	2 294
7	1 014	11 924	131 856	7 024
8	2 427	34 100	445 000	20 656
9	5 737	94 584	1 433 294	58 842
10	13 412	255 852	4 444 006	163 250
11	31 088	677 850	13 349 510	443 062
12	71 506	1 764 482	39 041 224	1 180 156
13	163 378	4 523 924	111 583 236	3 092 964
14	371 272	11 447 870	312 618 368	7 993 116
15	839 248	28 636 218	860 662 498	20 401 250
16	1 889 019	70 907 326	2 333 112 020	51 502 616
17	4 235 082	173 991 368	6 238 124 024	128 748 512
18	9 459 687	423 469 988	16 474 149 036	319 010 540
19	21 067 566	1 023 162 920	43 023 953 304	784 179 992
20	46 769 977	2 455 645 268	111 230 237 224	1 913 668 608
21	103 574 916	5 858 183 260	284 926 172 100	4 639 155 964
22	228 808 544	13 898 041 838	723 731 637 254	11 178 566 462
23	504 286 803	32 804 047 708	1 824 124 911 010	26 784 974 870
24	1109 344 029	77 067 740 230	4 564 862 407 124	63 851 541 584
25	2435 398 781	180 271 746 166	11 348 210 517 840	151 484 343 212
Triangular site percolation				
0	1	0	0	0
1	3	4	6	2
2	7	20	60	12
3	15	68	314	46
4	31	196	1 240	144
5	62	512	4 166	402
6	122	1 256	12 600	1 040
7	235	2 936	35 324	2 548
8	448	6 628	93 576	5 992
9	842	14 528	236 944	13 632
10	1 572	31 140	578 764	30 220
11	2 904	65 414	1 371 478	65 486
12	5 341	135 276	3 169 380	139 404
13	9 743	275 656	7 165 478	291 770
14	17 718	555 216	15 901 324	602 908
15	32 009	1 105 726	34 705 018	1 229 242
16	57 701	2 182 380	74 661 832	2 482 792
17	103 445	4 268 906	158 529 158	4 959 014
18	185 165	8 290 740	332 756 408	9 836 840
19	329 904	15 984 420	691 084 378	19 323 246
20	587 136	30 638 312	1 421 836 528	37 773 464
21	1 040 674	58 369 924	2 899 678 894	73 182 570
22	1 843 300	110 665 328	5 867 341 452	141 345 292
23	3 253 020	208 734 268	11 784 640 984	270 647 584
24	5 738 329	392 103 508	23 512 608 484	517 513 972
25	10 090 036	733 311 754	46 616 228 682	980 893 354
26	17 736 533	1366 650 536	91 894 597 756	1859 946 412

3. Analysis of series

We have analysed the series using inhomogeneous differential approximants in the manner described by Guttmann (1987). This method is intrinsically superior to the standard Dlog Padé method for such series, as the latter method cannot accommodate additive analytic terms, as discussed above. Such terms slow the convergence of the Padé method. First-order inhomogeneous approximants can include such additive terms, while second-order approximants can additionally include a confluent singularity. It is commonly found that first-order approximants provide more stable estimates of the critical parameters than do second-order approximants, even when confluent terms are believed to be present. Such effects are due either to the weakness of the confluent term, or to the fact that unrealistically long series are usually required to detect the presence of such confluent terms. In any event, for the directed percolation problem, the correction to scaling exponent is believed to be very close to 1 (see I) and as such would be effectively indistinguishable from an analytic correction. Further evidence for the absence of a correction to scaling exponent is given in a recent paper by Baxter and Guttmann (1988). For all the above reasons then, we have based our analysis on first-order differential approximants only. It is fair to say that, despite the claimed superiority of differential approximants, the results we have obtained for the square lattice bond problem are no better than those obtained in I. The estimates for the triangular lattice series for both the site and bond problem are, however, new, as are the results for the square lattice site problem. These results provide additional evidence in support of the conjectured exponent values.

We first analysed the mean-size series for the bond and site problem on the square and triangular lattice. The results of our analysis are shown in tables 2 and 3. The method of analysis is described in Guttmann (1987). For a given number of series coefficients, inhomogeneous first-order differential approximants $[L/N+\Lambda;\,N]$, $\Lambda=-1,0,1$ are formed, with L, the degree of the inhomogeneous polynomial, ranging from 1 to 8, or 0 to 10. Non-defective approximants are then used to give mean values of the exponent and critical point. These are defined to be approximants with no singularity, other than the physical singularity, in that region of the complex plane defined by

$$|\mathrm{Im}(z)|<0.005 \qquad 0.0<\mathrm{Re}(z)<1.15z_c \tag{3.1}$$

where z is the expansion variable of the series, and z_c is the critical point, or in this case the percolation probability.

In table 2 we show some of the exponent and critical point estimates for the triangular lattice bond problem mean-size series, with L, the degree of the inhomogeneous polynomial ranging from 1 to 4. Similar tables were constructed for the other three series (triangular site, square site and square bond) but to save space we present only a summary of these data in table 3. Thus in table 3 we list the means, quoting an error equal to two standard deviations. The last column shows the number, l, of approximants used in forming the estimates, that is, defective approximants are not included, while the first column gives the number, n, of series coefficients used in forming the approximant. For the triangular lattice site problem, p_c and γ are steadily increasing. It is very difficult to judge the limit of these sequences, but a value of $2.7777\ldots$ for γ seems entirely attainable. For the triangular lattice bond problem, the estimates are not monotonic, but there is a general upward trend, which has taken the estimate of γ slightly above $2.7777\ldots$, but with error bars that encompass this value.

Table 2. First-order differential approximants to the mean-size series of the triangular lattice bond problem. Defective approximants are marked with an asterisk.

Number of approximant								
4	5	6	7	8	9	10	11	12
$L=1$								
0.477 7261	—	—	0.478 1210	0.478 6385	0.478 2323	0.478 1625	0.478 0377	0.478 0240
−2.256 956*	—	—	−2.285 832	−2.338 980	−2.296 832*	−2.302 808*	−2.280 113	−2.278 272
—	0.478 5941	0.478 5864	0.478 4830	0.478 1378	0.478 0517	0.478 1666	0.478 0297	
—		−2.321 744	−2.326 006	−2.318 421	−2.288 102*	−2.280 845	−2.303 685*	−2.279 025*
0.477 8185	0.478 2380	0.478 4554	0.478 3129	0.478 1519	0.477 9671	0.477 9860	0.478 0311	
−2.270 819	−2.296 048	−2.315 881	−2.303 876	−2.289 576*	−2.272 552*	−2.274 521*	−2.279 281	
$L=2$								
0.477 7334	0.478 4094	—	0.477 8496	0.478 1087	0.478 0242	0.478 4531	0.478 0242	
−2.263 758	−2.303 783	—	−2.266 812	−2.286 115	−2.278 398	−2.409 047*	−2.278 373	
0.477 6218	—	0.478 2818	0.478 1876	0.478 1273	0.478 0230	0.478 0295	0.478 0281	
−2.258 692	—	−2.298 495	−2.293 132	−2.287 418*	−2.278 292	−2.279 048	−2.278 832	
—	—	0.478 4641	0.478 0947	0.477 9219	0.478 0248	0.478 0266		
—	—	−2.316 743	−2.285 057	−2.268 856	−2.278 472	−2.278 659		
$L=3$								
0.477 5406	0.478 0120	0.478 1400	0.478 0317	0.477 9801	0.478 0180	0.478 0283	0.478 0287	
−2.254 600	−2.278 681	−2.286 920	−2.278 708	−2.273 599*	−2.277 716	−2.278 898	−2.278 968	
0.477 9907	—	0.478 0035	0.478 0516	0.477 9899	0.478 0287	0.478 0273		
−2.277 471	—	−2.276 845	−2.280 496	−2.274 538*	−2.278 946	−2.278 788		
0.477 8905	0.477 3235	0.478 0768	0.478 0315	0.477 9938	0.478 0284	0.478 0304		
−2.270 916*	−2.241 914*	−2.282 781	−2.278 472*	−2.274 853*	−2.278 910	−2.279 311*		
$L=4$								
0.478 0015	0.477 8674	0.478 0733	0.478 1013	0.477 9935	0.478 0174	0.478 0267		
−2.278 263	−2.270 484*	−2.283 055	−2.285 232	−2.274 968*	−2.277 652	−2.278 717		
0.477 9687	0.478 0981	0.478 1449	0.477 4186	0.478 0172	0.478 1740	0.478 0281		
−2.275 930	−2.284 401	−2.289 117	−2.194 452*	−2.277 632	−2.301 582	−2.278 882		
0.478 1256	0.478 0443	0.478 2112	0.477 9764	0.478 0151	0.478 0267			
−2.286 283	−2.280 506	−2.294 595	−2.272 875*	−2.277 390	−2.278 709			

Table 3. Results of the analysis of the mean-size series by first-order differential approximants. See text for explanation of n and l.

n	p_c	γ	l
Triangular lattice site problem			
20	0.599 530 (637)	2.2700 (372)	10
21	0.595 575 (67)	2.2703 (60)	9
22	0.595 582 (45)	2.2711 (46)	10
23	0.595 620 (30)	2.2749 (39)	8
24	0.595 627 (32)	2.2754 (39)	12
25	0.595 632 (69)	2.2761 (80)	12
26	0.595 633 (66)	2.2763 (86)	13
Triangular lattice bond problem			
19	0.478 082 (168)	2.2833 (152)	10
20	0.478 009 (66)	2.2767 (64)	15
21	0.478 010 (38)	2.2768 (40)	11
22	0.478 018 (14)	2.2777 (16)	10
23	0.478 024 (23)	2.2785 (26)	13
24	0.478 026 (7)	2.2786 (10)	15
25	0.478 025 (7)	2.285 (11)	11
Square lattice site problem			
40	0.705 503 (70)	2.2805 (106)	5
41	0.705 516 (41)	2.2831 (71)	3
42	0.705 515 (32)	2.2829 (55)	2
43	0.705 500 (59)	2.2802 (108)	5
44	0.705 507 (21)	2.2813 (41)	7
45	0.705 500 (37)	2.2797 (73)	3
46	0.705 504 (49)	2.2807 (96)	7
47	0.705 489 (21)	2.2778 (43)	7
48	0.705 491 (11)	2.2781 (22)	7
Square lattice bond problem			
40	0.644 696 (2)	2.2767 (2)	3
41	0.644 696 (2)	2.2766 (3)	11
42	0.644 695 (1)	2.2764 (2)	8
43	0.644 696 (4)	2.2767 (8)	9
44	0.644 695 (3)	2.2767 (9)	9
45	0.644 696 (2)	2.2767 (3)	5
46	0.644 697 (1)	2.2769 (3)	9
47	0.644 697 (3)	2.2769 (6)	17
48	0.644 698 (5)	2.2771 (11)	12
49	0.644 698 (3)	2.2771 (7)	11

Combining the results for the bond problem in the manner discussed in Guttmann (1987), which weights entries according to their associated error, gives the composite result

$$p_c = 0.478\,023 \pm 0.000\,005 \qquad \gamma = 2.2782 \pm 0.0007.$$

For the square lattice site problem, the results are generally trending downward, and a limit around 2.278 appears entirely attainable. The results for the square lattice bond problem are seen to be steadily increasing, and a limit around 2.278 is estimated. The difficulty in extrapolating these trends is that the nature of the convergence has been found not to be uniform (Guttmann 1988). Rather, it is found that trends continue until, at a certain value of n (the number of series coefficients used in forming the estimates), the estimates of the critical parameters stabilise. It is as if a certain number of terms is needed to successfully represent the function. Below this number, we get increasingly good estimates of the critical parameters as the number of terms increases.

For all four series a value of $\gamma = 2.278 \pm 0.002$ is consistent with our results. If we now make the assumption that this exponent is represented by a 'simple' rational fraction, where by 'simple' we mean a fraction with a denominator less than 100, we are immediately led to 41/18. Tentatively accepting this value, we obtain estimates of p_c for all four series by linear regression on the estimates used to give the results in table 3 (as described in Guttmann (1987)). In this way we find

$$p_c = 0.595\,646 \pm 0.000\,003 \qquad \text{triangular site problem}$$

$$p_c = 0.478\,018 \pm 0.000\,002 \qquad \text{triangular bond problem}$$

$$p_c = 0.705\,489 \pm 0.000\,004 \qquad \text{square site problem}$$

$$p_c = 0.644\,701 \pm 0.000\,001 \qquad \text{square bond problem.}$$

To analyse the first- and second-moment series, which are not as well behaved as the mean-size series, we fix the value of p_c to the value quoted above, and estimate the exponent from the biased differential approximants. In tables 4 and 5 we show typical biased estimates of the exponents for the square lattice bond problem zeroth- and second-moment series. Table 4 shows the estimate of the exponent for the mean-size series (the zeroth moment), and it is readily apparent that the exponent value $2.2777\ldots$ is well supported. Table 5 gives estimates of the second moment, $\langle x^2 \rangle$, exponent $\gamma + 2\nu_\perp$, for which we find the value 4.4716 ± 0.0003, while the corresponding table for $\gamma + 2\nu_\parallel$ (not shown) gives 5.7455 ± 0.0005. For the triangular lattice bond problem we find the corresponding biased estimates are $\gamma + 2\nu_\perp = 4.472 \pm 0.002$ and $\gamma + 2\nu_\parallel = 5.7455 \pm 0.002$ respectively. The error bars reflect the scatter of the estimates, but do not include errors associated with the uncertainty in p_c. For the square lattice bond problem, the possible error in p_c would only cause a variation of a few parts in the last place quoted, while for the less precise triangular lattice exponents, the corresponding error is no more than 1 in the last digit quoted.

These results, combined with our assumed result for γ, give $\nu_\perp = 1.0969 \pm 0.0003$ and $\nu_\parallel = 1.7339 \pm 0.0003$. The closest 'simple' fractions are $79/72 = 1.097\,222$ and $26/15 = 1.7333\ldots$ respectively. These sum to $\nu_\perp + \nu_\parallel = 2.8308 \pm 0.0006$. Using our numerical estimate of γ rather than the conjectured exact value gives $\nu_\perp + \nu_\parallel = 2.8306 \pm 0.0026$ for the sum. The corresponding results for the site problem are less well behaved. For the triangular lattice site problem we obtain $\gamma + 2\nu_\parallel = 5.745 \pm 0.005$ and $\gamma + 2\nu_\perp = 4.473 \pm 0.003$ respectively. These estimates are consistent with those quoted above, though they are of lower precision. The sum of the correlation function exponents is still 2.831. For the square lattice site problem, the precision is lower still. We find $\gamma + 2\nu_\parallel = 5.743 \pm 0.010$ and $\gamma + 2\nu_\perp = 4.471 \pm 0.007$. These results are therefore consistent with, but of lower precision than, those for the bond problem. This observation is also true for the 'ordinary', i.e. non-directed, percolation problem.

Taking the above estimates $\gamma = 2.278 \pm 0.002$ and $\nu_\perp + \nu_\parallel = 2.8306 \pm 0.0026$, scaling gives $\beta = 0.277 \pm 0.002$, and $\alpha = -0.831 \pm 0.002$. These values are just consistent with the fractions $\beta = 5/18 = 0.2777\ldots$ and $\nu_\perp + \nu_\parallel = 2\frac{5}{6} = 2.8333\ldots$ cited in § 1. If, however, we stick to the conjectured value of $\gamma = 2.2777\ldots$, then we obtain $\beta = 0.2765 \pm 0.0003$ and $\alpha = -0.8308 \pm 0.0003$. We note that $199/720 = 0.276\,38\ldots$ and $-299/360 =$

Table 4. Biased estimates of the exponent of the mean-size series for the square lattice bond problem.

Number of approximant										
14	15	16	17	18	19	20	21	22	23	24
$L=1$										
−2.277 242	−2.277 228	−2.277 229*	−2.277 230	−2.277 283	−2.277 116*	−2.278 883*	−2.277 319*	−2.276 960	−2.277 442	−2.277 634*
−2.277 220	−2.277 222	−2.277 222	−2.277 182*	−2.277 006*	−2.277 176	−2.276 976*	−2.276 862	−2.276 886	−2.277 378*	−2.277 492
−2.277 238	−2.277 212	−2.276 942*	−2.277 025*	−2.277 762	−2.277 188	−2.277 275	−2.276 941	−2.277 406*	−2.279 162*	
$L=2$										
−2.277 145	−2.277 241*	−2.277 224	−2.277 171*	−2.276 802*	−2.278 245*	−2.276 941*	−2.277 505*	−2.276 890*	−2.277 658	−2.277 787*
−2.277 225	−2.277 218	−2.277 205*	−2.277 187*	−2.276 888*	−2.277 291	−2.277 041*	−2.276 951	−2.276 872	−2.277 688*	
−2.277 211	−2.277 245	−2.277 178*	−2.277 010*	−2.276 985*	−2.277 022*	−2.277 409	−2.276 883	−2.277 479*	−2.277 603	
$L=3$										
−2.277 234	−2.277 218	−2.277 213*	−2.277 244	−2.276 430*	−2.276 864*	−2.277 664*	−2.277 838*	−2.277 763*	−2.277 706*	
−2.277 216	−2.277 219*	−2.277 203*	−2.277 160*	−2.277 813*	−2.277 420	−2.278 076*	−2.277 841*	−2.277 817*	−2.277 713*	
−2.277 218	−2.277 209*	−2.277 972*	−2.277 073*	−2.277 248	−2.276 555*	−2.277 766*	−2.277 719*	−2.277 733*		
$L=4$										
−2.277 251	−2.277 197	−2.277 315	−2.277 193*	−2.296 712*	−2.277 455	−2.277 834*	−2.277 837*	−2.277 790*	−2.266 519*	
−2.277 205	−2.277 225*	−2.277 271	−2.277 139*	−2.277 320	−2.277 369*	−2.277 908*	−2.277 728*	−2.277 777*		
−2.277 198	−2.277 162*	−2.277 140*	−2.276 991*	−2.277 526*	−2.279 707*	−2.277 696*	−2.277 682*	−2.277 765*		
$L=5$										
−2.277 261	−2.276 780*	−2.277 311	−2.278 435	−2.277 601	−2.277 766	−2.277 808*	−2.277 744	−2.277 779*		
−2.277 199	−2.277 296*	−2.276 801*	−2.277 491	−2.279 658*	−2.277 769	−2.277 783*	−2.277 751	−2.277 772*		
−2.277 186	−2.276 988*	−2.277 142*	−2.277 748	−2.278 016	−2.277 641	−2.277 732*	−2.277 788			

Table 4. (continued)

Number of approximant										
14	15	16	17	18	19	20	21	22	23	24
$L=6$										
−2.277 092*	−2.279 752*	−2.316 083*	−2.277 564*	−2.277 743*	−2.277 770	−2.277 838*	−2.277 762	−2.277 770*		
−2.277 100*	−2.277 134*	−2.277 792*	−2.277 457*	−2.277 389*	−2.277 714*	−2.277 764	−2.277 770*			
−2.277 213	−2.276 904*	−2.277 522*	−2.277 998*	−2.277 720*	−2.277 684*	−2.277 763	−2.277 770*			
$L=7$										
−2.277 099*	−2.276 651*	−2.277 818*	−2.278 854*	−2.277 657*	−2.277 689*	−2.277 748	−2.277 759*			
−2.277 172*	−2.277 429*	−2.277 736*	−2.277 914*	−2.276 852*	−2.277 693*	−2.277 762	−2.277 770*			
−2.277 221	−2.277 839*	−2.277 017*	−2.278 113*	−2.277 711*	−2.277 704*	−2.277 715*				
$L=8$										
−2.277 115*	−2.277 343*	−2.277 585*	−2.277 834*	−2.277 769*	−2.277 668*	−2.277 773	−2.277 784*			
−2.277 123*	−2.276 236*	−2.277 582*	−2.277 468*	−2.277 548*	−2.277 920*	−2.277 776*				
−2.277 727*	−2.277 554*	−2.277 779*	−2.279 011*	−2.282 763*	−2.277 794*	−2.277 780*				
$L=9$										
−2.277 608	−2.277 874*	−2.277 589*	−2.277 613*	−2.277 647*	−2.277 746	−2.277 771				
−2.277 746*	−2.277 679	−2.278 609*	−2.277 607*	−2.277 609*	−2.277 769	−2.277 781*				
−2.277 750	−2.277 480*	−2.277 586*	−2.277 673*	−2.278 041*	−2.277 760					
$L=10$										
−2.277 745*	−2.277 778*	−2.277 654*	−2.277 629*	−2.266 058*	−2.277 758	−2.277 793*				
−2.277 745*	−2.278 355*	−2.277 645*	−2.277 666*	−2.276 934*	−2.277 802*					
−2.277 747*	−2.277 562*	−2.277 666*	−2.277 655*	−2.277 878*	−2.277 796*					

Table 5. Biased estimates of the exponent for the second-moment series $\mu_{2,0}$ for the square lattice bond problem.

Number of approximant										
14	15	16	17	18	19	20	21	22	23	24
$L=1$										
−4.472 646	−4.472 155*	−4.448 897*	−4.471 649	−4.471 710	−4.471 718	−4.471 709	−4.471 716	−4.471 688	−4.471 667	−4.471 618
−4.472 081*	−4.471 850*	−4.471 646	−4.471 670	−4.471 709*	−4.471 714	−4.471 652	−4.471 697	−4.471 704*	−4.471 667	
−4.472 535*	−4.471 782*	−4.471 641	−4.471 712	−4.471 720	−4.471 706	−4.471 734*	−4.471 690	−4.471 667	−4.471 667*	
$L=2$										
−4.471 936*	−4.471 473	−4.492 182*	−4.471 690	−4.471 757	−4.471 717	−4.471 721*	−4.471 688	−4.471 692*	−4.471 613	
−4.472 063*	−4.897 072*	−4.472 650	−4.471 663	−4.471 717	−4.471 713	−4.471 704	−4.471 684	−4.471 710*	−4.471 631	
−4.471 768*	−4.471 511*	−4.471 673	−4.471 601*	−4.471 718	−4.471 729*	−4.471 692	−4.471 694*	−4.471 650		
$L=3$										
−4.473 667*	−4.469 655*	4.471 376*	−4.471 794	−4.471 583*	−4.471 592*	−4.471 600*	−4.471 690	−4.471 641	−4.471 579*	
−4.472 690*	−4.471 394*	−4.472 340*	−4.471 875	−4.471 719	−4.471 671	−4.471 682	−4.471 753*	−4.471 627*		
−4.470 393*	−4.471 408*	−4.471 717	−4.471 774*	−4.471 749*	−4.471 633*	−4.471 698	−4.471 656	−4.471 640		
$L=4$										
−4.470 152*	−4.472 236	−4.474 245*	−4.471 831	−4.471 728	−4.471 647	−4.471 644	−4.471 645	−4.471 647*		
−4.471 868	−4.472 452*	−4.470 886*	−4.471 730	−4.471 715*	−4.471 615*	−4.471 658	−4.471 645	−4.471 642*		
−4.472 254*	−4.474 359*	−4.471 786	−4.471 729	−4.471 891*	−4.471 801*	−4.471 638*	−4.471 643*			
$L=5$										
−4.472 619*	−4.472 226	−4.472 114	−4.471 993	−4.471 656*	−4.471 647	−4.471 648	−4.471 645	−4.471 643*		
−4.472 294	−4.472 305*	−4.471 911	−4.471 728	−4.471 649*	−4.471 660	−4.471 644*	−4.471 642*			
−4.472 358*	−4.472 248	−4.473 230*	−4.471 695*	−4.479 682*	−4.471 607*	−4.471 646	−4.471 643*			

Table 5. (continued)

Number of approximant										
14	15	16	17	18	19	20	21	22	23	24
$L=6$										
−4.470 030*	−4.472 413*	−4.472 100*	−4.471 501*	−4.471 649*	−4.471 648	−4.471 640	−4.471 646*			
−4.472 150	−4.472 116	−4.471 859	−4.471 565*	−4.471 651	−4.471 643	−4.471 646*	−4.471 643*			
−4.472 798*	−4.471 792	−4.471 657	−4.471 650	−4.471 604*	−4.471 641	−4.471 646*				
$L=7$										
−4.472 198	−4.471 595	−4.471 665	−4.471 515*	−4.471 578*	−4.471 635	−4.471 640	−4.471 677*			
−4.471 912	−4.471 781	−4.471 467*	−4.471 733*	−4.471 618	−4.471 640	−4.471 643				
−4.471 733	−4.471 787	−4.471 600	−4.471 635	−4.471 621*	−4.471 630*	−4.472 411*				
$L=8$										
−4.472 677*	−4.471 805	−4.471 728*	−4.471 682*	−4.471 558*	−4.471 638	−4.471 640				
−4.471 713	−4.471 710*	−4.471 613*	−4.471 556*	−4.469 471*	−4.471 646	−4.472 268*				
−4.471 762	−4.471 695*	−4.471 641*	−4.471 534*	−4.471 631	−4.471 619*					
$L=9$										
−4.471 755*	−4.471 946*	−4.471 627	−4.471 642	−4.472 054*	−4.471 643	−4.471 640				
−4.471 767*	−4.471 700*	−4.471 651	−4.471 546*	−4.471 583*	−4.471 640*					
−4.471 758	−4.471 465*	−4.471 655	−4.471 189*	−4.471 640*	−4.471 632					
$L=10$										
−4.471 771*	−4.471 706*	−4.471 655*	−4.471 606*	−4.471 624	−4.471 638					
−4.471 788	−4.471 704*	−4.471 647	−4.471 556*	−4.471 638	−4.471 673*					
−4.471 693*	−4.471 660*	−4.471 176*	−4.471 642*	−4.471 640*						

$-0.8305\ldots$ and that these fractions follow from the conjectured values and the scaling relations.

Despite the apparent precision of the second-moment exponent estimates, it is worth remembering that such composite series, by which we mean series that depend on more than one exponent, are generally considerably less reliable than those series that are characterised by a single exponent. Thus while the errors quoted above do reflect the self-consistency of the exponent estimates, it would be a mistake to interpret them as absolute bounds. This is seen, for example, in the series for the square end-to-end distance in self-avoiding walks. That composite series diverges with exponent $\gamma+2\nu$, and gives considerably less accurate exponent estimates than the walk generating function series which diverges with exponent γ (Guttmann 1987). Alternative analyses, such as forming the quotient series $\mu_{2,0}/S$ and $\mu_{0,2}/S$ should give series which diverge at p_c with exponents $2\nu_\perp$ and $2\nu_\|$ respectively. In this way we find identical estimates for the exponents to those quoted above, without using the estimate of γ.

Turning now to the first-moment $\langle t\rangle$ series, the exponent for this series is $\gamma+\nu_\|$, and biased estimates were obtained as for the second-moment series. These are

$$\gamma+\nu_\| = 4.008 \pm 0.002 \qquad \text{triangular site}$$

$$\gamma+\nu_\| = 4.0111 \pm 0.0003 \qquad \text{triangular bond}$$

$$\gamma+\nu_\| = 4.012 \pm 0.002 \qquad \text{square site}$$

$$\gamma+\nu_\| = 4.0115 \pm 0.0004 \qquad \text{square bond.}$$

Again we see that the bond problem estimates are more accurate than the site problem estimates, and combining these with the conjectured value of γ gives $\nu_\| = 1.7336 \pm 0.0006$, a result entirely consistent with that obtained from the second-moment series. We have also analysed the first-moment bond problem series without biasing, and obtain the following results:

$$p_c = 0.478\,025 \pm 0.000\,006 \qquad \gamma+\nu_\| = 4.013 \pm 0.0009 \qquad \text{triangular bond}$$

$$p_c = 0.644\,697 \pm 0.000\,006 \qquad \gamma+\nu_\| = 4.010 \pm 0.001 \qquad \text{square bond.}$$

These values are in complete accord with those quoted above, both from other series and from different analyses.

4. Discussion

As mentioned in the introduction, the theory of conformal invariance is not applicable to such non-translationally invariant problems as this. Nevertheless, it is perhaps interesting to look at the scaling indices for this problem to see if perchance they do correspond to a set of values characterised by a particular central charge. From the relation $2/y_T = 2-\alpha$ and $x_T + y_T = 2$, we obtain $x_T = 1318/1019$. It is clear that this does not correspond to any simple realisation of the Kac formula, or any reasonable value of the central charge. The same is true of the simpler set of exponents which we rejected.

In a recent paper (Baxter and Guttmann 1988) we have studied the percolation probability series, which gives a direct estimate of the exponent β. This supports the conjectured values quoted in § 1. It is hoped that the conjectured exact exponent set may help in the search for an exact solution. In conclusion we remark that the numerically close exponent set $\nu_\perp + \nu_\| = 17/6$, $\beta = 5/18$, $\alpha = -5/6$ and $\delta = 46/5$ is aesthetically far more satisfactory, but regrettably is not as well supported numerically as those to which we have reluctantly been led: $\gamma = 41/18$, $\nu_\perp = 79/72$, $\nu_\| = 26/15$, $\beta = 199/720$, $\alpha = -299/360$ and $\delta = 1839/199$. While we are sympathetic with the view that these horrible fractions appear far less likely than the numerically close set mentioned, the numerical evidence is firmly in favour of the values we have conjectured. The only additional comment we can offer is that perhaps such unappealing exponents are characteristic of directed problems.

Acknowledgments

AJG would like to thank John Cardy for introducing him to this problem, and the Australian Research Grants Scheme for financial support.

References

Adler J, Moshe M and Privman V 1981 *J. Phys. A: Math. Gen.* **14** L363
Baxter R J and Guttmann A J 1988 *J. Phys. A: Math. Gen.* **21** 3193
Bhatti F M and Essam J W 1984 *J. Phys. A: Math. Gen.* **17** L67
Blease J 1977 *J. Phys. C: Solid State Phys.* **10** 3461
Cardy J 1987 *Phase Transitions and Critical Phenomena* vol 11, ed C Domb and J Lebowitz (New York: Academic) pp 55-126
De'Bell K and Essam J W 1983 *J. Phys. A: Math. Gen.* **16** 385
Essam J W 1972 *Phase Transitions and Critical Phenomena* vol 2, ed C Domb and M S Green (New York: Academic) pp 197-270
Essam J W, De'Bell K and Adler J 1986 *Phys. Rev.* **33** 1982
Guttmann A J 1987 *J. Phys. A: Math. Gen.* **20** 1839
—— 1989 *Phase Transitions and Critical Phenomena* vol 12, ed C Domb and J Lebowitz (New York: Academic) to be published
Guttmann A J and Joyce G S 1972 *J. Phys. A: Gen. Phys.* **5** L81
Joyce G S and Guttmann A J 1973 *Padé Approximants and their Applications* ed P R Graves-Morris (New York: Academic) pp 163-8
Rehr J J, Joyce G S and Guttmann A J 1980 *J. Phys. A: Math. Gen.* **13** 1587